A SHORT TABLE OF INTEGRALS

COMPILED BY

B. O. PEIRCE

LATE HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY
IN HARVARD UNIVERSITY

ABRIDGED EDITION

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FUNDAMENTAL EQUATIONS

1. $\int a \cdot f(x)dx = a\int f(x)dx$; $\int \phi(y)dx = \int \frac{\phi(y)}{y'}dy$, where $y' = dy/dx$.

2. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$, where u and v are any functions of x.

3. $\int u\,dv = uv - \int v\,du$; $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.

4. $\int x^m dx = \frac{x^{m+1}}{m+1}$, if $m \neq -1$; $\int \frac{dx}{x} = \log x$, or $\log(-x)$.

5. $\int e^{ax}dx = e^{ax}/a$; $\int b^{ax}dx = \frac{b^{ax}}{a\log b}$.

6. $\int \sin x\,dx = -\cos x$; $\int \cos x\,dx = \sin x$.

$\int \tan x\,dx = -\log\cos x$; $\int \operatorname{ctn} x\,dx = \log\sin x$.

$\int \sec^2 x\,dx = \tan x$; $\int \csc^2 x\,dx = -\operatorname{ctn} x$.

7. $\int \cosh x\,dx = \sinh x$; $\int \sinh x\,dx = \cosh x$.

$\int \tanh x\,dx = \log\cosh x$; $\int \operatorname{ctnh} x = \log\sinh x$.

8. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$, or $-\frac{1}{a}\operatorname{ctn}^{-1}\left(\frac{x}{a}\right)$.

$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, or $\frac{1}{2a}\log\frac{a+x}{a-x}$.

$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\operatorname{ctnh}^{-1}\left(\frac{x}{a}\right)$, or $\frac{1}{2a}\log\frac{x-a}{x+a}$.

9. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \text{ or } -\cos^{-1}\left(\frac{x}{a}\right).$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right).$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right).$$

$$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right).$$

10. $$\int \frac{dx}{x\sqrt{a + bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a + bx}{-a}}, \text{ or } \frac{-2}{\sqrt{a}}\tanh^{-1}\sqrt{\frac{a + bx}{a}}.$$

In such a case as this, that one of the alternate values of the integral which makes the quantities under the radical signs positive is to be used, and each radical itself is to be considered positive. Of course an arbitrary constant may be added to the value of every integral given in this pamphlet.

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11. $e^{xi} = \cos x + i \sin x$; $e^{-xi} = \cos x - i \sin x$.

12. $\sinh x = \frac{1}{2}(e^x - e^{-x})$; $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

13. $\sin xi = i \sinh x$; $\cos xi = \cosh x$.

14. $\sin x = -i \sinh xi$; $\cos x = \cosh xi$.

15. $\log u = \log(cu) - \log c$.

16. $\log x = \log(-x) + (2k + 1)\pi i$; $\log_e x = (2.3025851) \cdot \log_{10} x$.

17. $\log(x \pm yi) = \frac{1}{2}\log(x^2 + y^2) \pm i \tan^{-1}(y/x)$.

For acute angles and some other cases easily to be determined in each instance,

18. $\sin^{-1}u = \cos^{-1}\sqrt{1 - u^2} = \tan^{-1}\left(u/\sqrt{1 - u^2}\right) = \csc^{-1}(1/u)$.

19. $\sin^{-1}u = -\sin^{-1}\sqrt{1 - u^2} +$ a constant $= \frac{1}{2}\sin^{-1}(2u^2 - 1) +$ a constant.

20. $\tan^{-1}u = -\tan^{-1}(1/u) +$ a constant.

I. RATIONAL ALGEBRAIC FUNCTIONS

A. Expressions involving $(a + bx)$

The substitution of y or z for x, where $y = xz = a + bx$, gives

21. $$\int (a + bx)^m dx = \frac{1}{b}\int y^m dy.$$

22. $$\int x(a + bx)^m dx = \frac{1}{b^2}\int y^m (y - a)\, dy.$$

23. $$\int x^n (a + bx)^m dx = \frac{1}{b^{n+1}}\int y^m (y - a)^n dy.$$

24. $$\int \frac{x^n dx}{(a + bx)^m} = \frac{1}{b^{n+1}}\int \frac{(y - a)^n dy}{y^m}.$$

25. $$\int \frac{dx}{x^n (a + bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(z - b)^{m+n-2} dz}{z^m}.$$

Whence

26. $$\int \frac{dx}{a + bx} = \frac{1}{b}\log(a + bx).$$

27. $$\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}.$$

28. $$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}.$$

29. $$\int \frac{xdx}{a + bx} = \frac{1}{b^2}[a + bx - a\log(a + bx)].$$

30. $$\int \frac{xdx}{(a + bx)^2} = \frac{1}{b^2}\left[\log(a + bx) + \frac{a}{a + bx}\right].$$

31. $$\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$$

32. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right].$$

33. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx - 2a\log(a+bx) - \frac{a^2}{a+bx}\right].$$

34. $$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$$

35. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}.$$

36. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$$

B. Expressions involving $(a + bx^n)$

37. $$\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c} = \frac{1}{c}\sin^{-1}\frac{x}{\sqrt{c^2+x^2}}.$$

38. $$\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}; \quad \int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}.$$

$$\int \frac{dx}{c^2-x^2} = \frac{1}{c}\tanh^{-1}\left(\frac{x}{c}\right); \quad \int \frac{dx}{x^2-c^2} = -\frac{1}{c}\operatorname{ctnh}^{-1}\left(\frac{x}{c}\right).$$

39. $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right), \ [a>0,\ b>0].$$

40. $$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}},$$

or $$\frac{1}{\sqrt{-ab}}\tanh^{-1}\left(x\sqrt{\frac{-b}{a}}\right), \ [a>0,\ b<0].$$

41. $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}.$$

42. $$\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2\,ma}\frac{x}{(a+bx^2)^m} + \frac{2\,m-1}{2\,ma}\int \frac{dx}{(a+bx^2)^m}.$$

43. $$\int \frac{xdx}{a+bx^2} = \frac{1}{2\,b}\log\left(x^2+\frac{a}{b}\right).$$

44. $$\int \frac{xdx}{(a+bx^2)^{m+1}} = \frac{1}{2}\int \frac{dz}{(a+bz)^{m+1}},\ [z=x^2].$$

45. $$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2\,a}\log\frac{x^2}{a+bx^2}.$$

46. $$\int \frac{x^2dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}.$$

47. $$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}.$$

48. $$\int \frac{x^2dx}{(a+bx^2)^{m+1}} = \frac{-x}{2\,mb(a+bx^2)^m} + \frac{1}{2\,mb}\int \frac{dx}{(a+bx^2)^m}.$$

49. $$\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}.$$

50. $$\int \frac{dx}{a+bx^3} = \frac{k}{3\,a}\left[\frac{1}{2}\log\frac{(k+x)^2}{k^2-kx+x^2} + \sqrt{3}\tan^{-1}\frac{2\,x-k}{k\sqrt{3}}\right],\ [bk^3=a].$$

51. $$\int \frac{xdx}{a+bx^3} = \frac{1}{3\,bk}\left[\frac{1}{2}\log\frac{k^2-kx+x^2}{(k+x)^2} + \sqrt{3}\tan^{-1}\frac{2\,x-k}{k\sqrt{3}}\right],\ [bk^3=a].$$

52. $$\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log\frac{x^n}{a+bx^n}.$$

53. $$\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^ndx}{(a+bx^n)^{m+1}}.$$

54. $$\int \frac{x^mdx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}dx}{(a+bx^n)^{p+1}}.$$

55. $$\int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}.$$

56. $$\int x^{m-1}(a+bx^n)^p\,dx = \begin{cases} \dfrac{1}{b(m+np)}\left[x^{m-n}(a+bx^n)^{p+1}-(m-n)a\displaystyle\int x^{m-n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{m+np}\left[x^m(a+bx^n)^p+npa\displaystyle\int x^{m-1}(a+bx^n)^{p-1}\,dx\right]. \\ \dfrac{1}{ma}\left[x^m(a+bx^n)^{p+1}-(m+np+n)b\displaystyle\int x^{m+n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1}+(m+np+n)\displaystyle\int x^{m-1}(a+bx^n)^{p+1}\,dx\right]. \end{cases}$$

C. Expressions involving $(a+bx+cx^2)$

Let $X = a+bx+cx^2$ and $q = 4ac-b^2$, then

57. $$\int\frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\left(\frac{2cx+b}{\sqrt{q}}\right), \text{ when } q>0,$$

or $$\frac{-2}{\sqrt{-q}}\cdot\tanh^{-1}\left(\frac{2cx+b}{\sqrt{-q}}\right), \text{ when } q<0.$$

58. $$\int\frac{dx}{X} = \frac{1}{\sqrt{-q}}\log\frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \text{ when } q<0.$$

59. $$\int\frac{dx}{X^2} = \frac{2cx+b}{qX}+\frac{2c}{q}\int\frac{dx}{X}.$$

60. $$\int\frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2}+\frac{3c}{qX}\right)+\frac{6c^2}{q^2}\int\frac{dx}{X}.$$

61. $$\int\frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n}+\frac{2(2n-1)c}{qn}\int\frac{dx}{X^n}.$$

62. $$\int\frac{xdx}{X}=\frac{1}{2c}\log X-\frac{b}{2c}\int\frac{dx}{X}.$$

63. $$\int\frac{xdx}{X^2}=-\frac{bx+2a}{qX}-\frac{b}{q}\int\frac{dx}{X}.$$

64. $$\int\frac{xdx}{X^{n+1}}=-\frac{2a+bx}{nqX^n}-\frac{b(2n-1)}{nq}\int\frac{dx}{X^n}.$$

65. $$\int\frac{x^2}{X}dx=\frac{x}{c}-\frac{b}{2c^2}\log X+\frac{b^2-2ac}{2c^2}\int\frac{dx}{X}.$$

66. $$\int\frac{x^2}{X^2}dx=\frac{(b^2-2ac)x+ab}{cqX}+\frac{2a}{q}\int\frac{dx}{X}.$$

67. $$\int\frac{x^m dx}{X^{n+1}}=-\frac{x^{m-1}}{(2n-m+1)cX^n}-\frac{n-m+1}{2n-m+1}\cdot\frac{b}{c}\int\frac{x^{m-1}dx}{X^{n+1}}$$
$$+\frac{m-1}{2n-m+1}\cdot\frac{a}{c}\int\frac{x^{m-2}dx}{X^{n+1}}.$$

68. $$\int\frac{ax}{xX}=\frac{1}{2a}\log\frac{x^2}{X}-\frac{b}{2a}\int\frac{dx}{X}.$$

69. $$\int\frac{dx}{x^2X}=\frac{b}{2a^2}\log\frac{X}{x^2}-\frac{1}{ax}+\left(\frac{b^2}{2a^2}-\frac{c}{a}\right)\int\frac{dx}{X}.$$

70. $$\int\frac{dx}{x^mX^{n+1}}=-\frac{1}{(m-1)ax^{m-1}X^n}-\frac{n+m-1}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}}$$
$$-\frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}}.$$

D. Rational Fractions

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)}=\frac{g_1x^{n-1}+g_2x^{n-2}+g_3x^{n-3}+\cdots+g_n}{x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n}.$$

If a, b, c, etc. are the roots of the equation $F(x)=0$, so that

$$F(x)=(x-a)^p(x-b)^q(x-c)^r\cdots,$$

then
$$\begin{aligned}\frac{f(x)}{F(x)}=&\frac{A_1}{(x-a)^p}+\frac{A_2}{(x-a)^{p-1}}+\frac{A_3}{(x-a)^{p-2}}+\cdots+\frac{A_p}{x-a}\\&+\frac{B_1}{(x-b)^q}+\frac{B_2}{(x-b)^{q-1}}+\frac{B_3}{(x-b)^{q-2}}+\cdots+\frac{B_q}{x-b}\\&+\frac{C_1}{(x-c)^r}+\frac{C_2}{(x-c)^{r-1}}+\frac{C_3}{(x-c)^{r-2}}+\cdots+\frac{C_r}{x-c}\\&+\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots,\end{aligned}$$
where the numerators of the separate fractions are constants.

If a, b, c, etc. are single roots, then $p=q=r=\cdots=1$, and

$$\frac{f(x)}{F(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\cdots,$$

where
$$A=\frac{f(a)}{F'(a)},\quad B=\frac{f(b)}{F'(b)},\text{ etc.}$$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the following formulas:

71. $\displaystyle\int\frac{h\,dx}{(mx+n)^l}=\int\frac{h\,d(mx+n)}{m(mx+n)^l}=\frac{h}{m(1-l)(mx+n)^{l-1}}.$

72. $\displaystyle\int\frac{h\,dx}{mx+n}=\frac{h}{m}\log(mx+n).$

If any of the roots of the equation $f(x)=0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined, and the integral thus brought into a real form. The following formula, in which $i=\sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

73. $\log(x\pm yi)=\frac{1}{2}\log(x^2+y^2)\pm i\tan^{-1}\frac{y}{x}.$

II. IRRATIONAL ALGEBRAIC FUNCTIONS

A. Expressions involving $\sqrt{a+bx}$

The substitution of a new variable of integration, $y=\sqrt{a+bx}$, gives

74. $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$

75. $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

76. $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$

77. $\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$

78. $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$

79. $\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$

80. $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$

81. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right).$

82. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{-2}{\sqrt{a}}\tanh^{-1}\sqrt{\frac{a+bx}{a}}.$

83. $\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$

84. $\int (a+bx)^{\pm\frac{n}{2}}dx = \frac{2}{b}\int y^{1\pm n}dy = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}.$

85. $$\int x(a+bx)^{\pm\frac{n}{2}}dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$$

86. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int\frac{x^{m-1}dx}{\sqrt{a+bx}}.$$

87. $$\int\frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int\frac{dx}{x^{n-1}\sqrt{a+bx}}.$$

88. $$\int\frac{(a+bx)^{\frac{n}{2}}dx}{x} = b\int(a+bx)^{\frac{n-2}{2}}dx + a\int\frac{(a+bx)^{\frac{n-2}{2}}}{x}dx.$$

89. $$\int\frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int\frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int\frac{dx}{(a+bx)^{\frac{m}{2}}}.$$

B. Expressions involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$

90. $$\int\sqrt{x^2\pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\log\left(x+\sqrt{x^2\pm a^2}\right)\right].^*$$

91. $$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right].$$

92. $$\int\frac{dx}{\sqrt{x^2\pm a^2}} = \log\left(x+\sqrt{x^2\pm a^2}\right).^*$$

93. $$\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right),\ \text{or}\ -\cos^{-1}\left(\frac{x}{a}\right).$$

94. $$\int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right),\ \text{or}\ \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right).$$

95. $$\int\frac{dx}{x\sqrt{a^2\pm x^2}} = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right).^*$$

96. $$\int\frac{\sqrt{a^2\pm x^2}}{x}dx = \sqrt{a^2\pm x^2} - a\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right).^*$$

* $\log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) = \sinh^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) = \cosh^{-1}\left(\frac{x}{a}\right)$.

$\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) = \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$; $\log\left(\frac{a+\sqrt{a^2+x^2}}{x}\right) = \operatorname{csch}^{-1}\left(\frac{x}{a}\right)$.

97. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.$

98. $\int \frac{x\, dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2}.$

99. $\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$

100. $\int x \sqrt{x^2 \pm a^2}\, dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}.$

101. $\int x \sqrt{a^2 - x^2}\, dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}.$

102. $\int \sqrt{(x^2 \pm a^2)^3}\, dx$

$$= \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log\left(x + \sqrt{x^2 \pm a^2}\right)\right].^*$$

103. $\int \sqrt{(a^2 - x^2)^3}\, dx$

$$= \frac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{a}\right].$$

104. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}.$

105. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}.$

106. $\int \frac{x\, dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

107. $\int \frac{x\, dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$

108. $\int x \sqrt{(x^2 \pm a^2)^3}\, dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}.$

109. $\int x \sqrt{(a^2 - x^2)^3}\, dx = -\frac{1}{5} \sqrt{(a^2 - x^2)^5}.$

* See note on page 12

110. $\int x^2\sqrt{x^2 \pm a^2}\,dx$

$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right).^*$$

111. $\int x^2\sqrt{a^2 - x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right).$$

112. $\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right).^*$

113. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

114. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$

115. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$

116. $\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).^*$

117. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$

118. $\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).^*$

119. $\int \frac{x^2\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$

C. Expressions involving $\sqrt{a + bx + cx^2}$

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f\left(x, \sqrt{a + bx + cx^2}\right)$ we may put $\sqrt{a + bx + cx^2} = \sqrt{\pm c}\sqrt{A + Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z, such that

* See note on page 12

$z = \sqrt{A + Bx + x^2} - x$, if $c > 0$;

$z = \dfrac{\sqrt{A + Bx - x^2} - \sqrt{A}}{x}$, if $c < 0$ and $\dfrac{a}{-c} > 0$;

$z = \sqrt{\dfrac{x - \beta}{\alpha - x}}$, where α and β are the roots of the equation

$$A + Bx - x^2 = 0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

120. $\displaystyle\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right),$

or $\displaystyle\frac{1}{\sqrt{c}} \sinh^{-1}\left(\frac{2cx + b}{\sqrt{4ac - b^2}}\right)$, if $c > 0$.

121. $\displaystyle\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1}\left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right)$, if $c < 0$.

122. $\displaystyle\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}.$

123. $\displaystyle\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}\left(\frac{1}{X} + 2k\right).$

124. $\displaystyle\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx + b)\sqrt{X}}{(2n - 1)qX^n} + \frac{2k(n - 1)}{2n - 1}\int \frac{dx}{X^{n-1}\sqrt{X}}.$

125. $\displaystyle\int \sqrt{X}dx = \frac{(2cx + b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}.$

126. $\displaystyle\int X\sqrt{X}dx = \frac{(2cx + b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}.$

127. $\displaystyle\int X^2\sqrt{X}\,dx = \frac{(2cx + b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}.$

128. $\displaystyle\int X^n\sqrt{X}dx = \frac{(2cx + b)X^n\sqrt{X}}{4(n + 1)c} + \frac{2n + 1}{2(n + 1)k}\int \frac{X^n dx}{\sqrt{X}}.$

129. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}.$

130. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$.

131. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}$ [illegible]

132. $\int \frac{x^2dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int$ [illegible]

133. $\int \frac{x^2dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int$ [illegible]

134. $\int \frac{x^2dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac}{}$ [illegible]

135. $\int \frac{x^3dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X}$ [illegible]

136. $\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c}\int\sqrt{X}\,dx$ [illegible]

137. $\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X$ [illegible]

138. $\int \frac{xX^ndx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c}\int$ [illegible]

139. $\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c}$ [illegible]

140. $\int \frac{x^2X^ndx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2}{4}$ [illegible]

$-\frac{a}{2(n+1)c}\int\frac{X^ndx}{\sqrt{X}}$.

141. $\int x^3\sqrt{X}\,dx = \left(x^2 - \frac{7bx}{8c}\right.$ [illegible]

$+\left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3}\right)\int$ [illegible]

142. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right)$, if $a > 0$.

143. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{b^2-4ac}}\right)$, if $a < 0$.

144. $\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}$, if $a = 0$.

145. $\int \frac{dx}{xX^n\sqrt{X}} = \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a}\int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a}\int \frac{dx}{X^n\sqrt{X}}.$

146. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$

147. $\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$

148. $\int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a\int \frac{X^{n-1}dx}{x\sqrt{X}} + \frac{b}{2}\int \frac{X^{n-1}dx}{\sqrt{X}}.$

149. $\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$

150. $\int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}dx}{X^n\sqrt{X}}.$

151. $$\int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)}\int \frac{x^{m-1}X^n dx}{\sqrt{X}}$$
$$- \frac{(m-1)a}{(2n+m)c}\int \frac{x^{m-2}X^n dx}{\sqrt{X}}.$$

152. $$\int \frac{dx}{x^m X^n \sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n}$$
$$- \frac{(2n+2m-3)b}{2a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}}$$
$$- \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

130. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$$

131. $$\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}.$$

132. $$\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}.$$

133. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$$

134. $$\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}.$$

135. $$\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}.$$

136. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c}\int \sqrt{X}\,dx.$$

137. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx.$$

138. $$\int \frac{xX^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c}\int \frac{X^n\,dx}{\sqrt{X}}.$$

139. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int \sqrt{X}\,dx.$$

140. $$\int \frac{x^2X^n\,dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2n+3)b}{4(n+1)c}\int \frac{xX^n\,dx}{\sqrt{X}} - \frac{a}{2(n+1)c}\int \frac{X^n\,dx}{\sqrt{X}}.$$

141. $$\int x^3\sqrt{X}\,dx = \left(x^2 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c}\right)\frac{X\sqrt{X}}{5c} + \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3}\right)\int \sqrt{X}\,dx.$$

142. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right)$, if $a > 0$.

143. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{bx+2a}{x\sqrt{b^2-4ac}}\right)$, if $a < 0$.

144. $\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}$, if $a = 0$.

145. $\int \frac{dx}{xX^n\sqrt{X}} = \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a}\int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a}\int \frac{dx}{X^n\sqrt{X}}.$

146. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}.$

147. $\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$

148. $\int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a\int \frac{X^{n-1}dx}{x\sqrt{X}} + \frac{b}{2}\int \frac{X^{n-1}dx}{\sqrt{X}}.$

149. $\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$

150. $\int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c}\int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c}\int \frac{x^{m-1}dx}{X^n\sqrt{X}} - \frac{a}{c}\int \frac{x^{m-2}dx}{X^n\sqrt{X}}.$

151. $\int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)}\int \frac{x^{m-1}X^n dx}{\sqrt{X}}$

$$- \frac{(m-1)a}{(2n+m)c}\int \frac{x^{m-2}X^n dx}{\sqrt{X}}.$$

152. $\int \frac{dx}{x^m X^n\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)ax^{m-1}X^n}$

$$- \frac{(2n+2m-3)b}{2a(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}}$$

$$- \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}.$$

153. $$\int \frac{X^n dx}{x^m \sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)x^{m-1}} + \frac{(2n-1)b}{2(m-1)}\int \frac{X^{n-1}dx}{x^{m-1}\sqrt{X}} + \frac{(2n-1)c}{m-1}\int \frac{X^{n-1}dx}{X^{m-2}\sqrt{X}}.$$

154. $$\int \frac{dx}{(a'+b'x)\sqrt{X}} = \frac{1}{\sqrt{-h}}\tan^{-1}\frac{2h+m(a'+b'x)}{2b'\sqrt{-hX}},$$

or $$\frac{1}{\sqrt{h}}\log\frac{2h+m(a'+b'x)-2b'\sqrt{hX}}{a'+b'x},$$

where $m = bb' - 2a'c$ and $h = ab'^2 - a'bb' + ca'^2$.

If $h = 0$, the value of the integral is $-2b'\sqrt{X}/[m(a'+b'x)]$.

D. Miscellaneous Algebraic Expressions

155. $$\int \sqrt{2ax - x^2}\, dx = \tfrac{1}{2}[(x-a)\sqrt{2ax-x^2} + a^2\sin^{-1}(x-a)/a].$$

156. $$\int \frac{dx}{\sqrt{2ax-x^2}} = \cos^{-1}\left(\frac{a-x}{a}\right).$$

157. $$\int \frac{dx}{\sqrt{a+bx}\cdot\sqrt{a'+b'x}} = \frac{2}{\sqrt{-bb'}}\tan^{-1}\sqrt{\frac{-b'(a+bx)}{b(a'+b'x)}},$$

or $$\frac{2}{\sqrt{bb'}}\tanh^{-1}\sqrt{\frac{b'(a+bx)}{b(a'+b'x)}}.$$

158. $$\int \sqrt{(a+bx)(a'+b'x)}\, dx = \frac{k+2b\sqrt{a'+b'x}}{4bb'}\sqrt{(a+bx)(a'+b'x)} - \frac{k^2}{8bb'}\int \frac{dx}{\sqrt{a+bx}\cdot\sqrt{a'+b'x}}, \quad [k = ab' - a'b].$$

159. $$\int \sqrt{\frac{a'+b'x}{a+bx}}\, dx = \frac{\sqrt{a+bx}\cdot\sqrt{a'+b'x}}{b} - \frac{k}{2b}\int \frac{dx}{\sqrt{a+bx}\sqrt{a'+b'x}}.$$

160. $$\int \sqrt{\frac{1+x}{1-x}}\, dx = \sin^{-1}x - \sqrt{1-x^2}.$$

161. $$\int \sqrt{\frac{x+a}{x+b}}\,dx = \sqrt{(x+a)(x+b)} + (a-b)\log\left(\sqrt{x+a}+\sqrt{x+b}\right).$$

162. $$\int \frac{dx}{\sqrt{(x-a)(a'-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{a'-a}}.$$

163. $$\int \frac{(px+q)\,dx}{(x-a')(x-b')\sqrt{a+bx+cx^2}} = \frac{q+a'p}{a'-b'}\int \frac{dx}{(x-a')\sqrt{a+bx+cx^2}}$$
$$-\frac{q+b'p}{a'-b'}\int \frac{dx}{(x-b')\sqrt{a+bx+cx^2}}.$$

164. $$\int \frac{dx}{(a'+b'x)\sqrt{a+bx+cx^2}}$$
$$= \frac{1}{\sqrt{h}}\cdot\log\left(\frac{2h+m(a'+b'x)-2b'\sqrt{h(a+bx+cx^2)}}{a'+b'x}\right),$$

or $$\frac{1}{\sqrt{-h}}\cdot\tan^{-1}\left(\frac{2h+m(a'+b'x)}{2b'\sqrt{-h(a+bx+cx^2)}}\right),$$

where $$m = bb' - 2a'c \text{ and } h = ab'^2 - a'bb' + ca'^2.$$

165. $$\int f\left\{x, \sqrt[n]{\frac{a+bx}{a'+b'x}}\right\}dx$$
$$= n(a'b-ab')\int f\left\{\frac{a-a'z^n}{b'z^n-b}, z\right\}\cdot\frac{z^{n-1}\,dz}{(b'z^n-b)^2},$$

where $$z^n(a'+b'x) = a+bx.$$

166. $$\int f\left(x, \sqrt{a+bx+cx^2}\right)dx$$
$$= 2\int f\left(\frac{2\sqrt{a}\cdot z-b}{1-z^2}, \frac{z^2\sqrt{a}-bz+\sqrt{a}}{1-z^2}\right)\cdot\frac{z^2\sqrt{a}-bz+\sqrt{a}}{(1-z^2)^2}\,dz,$$

where $$xz+\sqrt{a} = \sqrt{a+bx+cx^2}.$$

III. TRANSCENDENTAL FUNCTIONS

167. $\int \sin x\,dx = -\cos x.$

168. $\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4}\sin 2x.$

169. $\int \sin^3 x\,dx = -\frac{1}{3}\cos x(\sin^2 x + 2).$

170. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

171. $\int \cos x\,dx = \sin x.$

172. $\int \cos^2 x\,dx = \frac{1}{2}\sin x\cos x + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4}\sin 2x.$

173. $\int \cos^3 x\,dx = \frac{1}{3}\sin x(\cos^2 x + 2).$

174. $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$

175. $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x.$

176. $\int \sin^2 x\cos^2 x\,dx = -\frac{1}{8}(\frac{1}{4}\sin 4x - x).$

177. $\int \sin x\cos^m x\,dx = -\frac{\cos^{m+1}x}{m+1}.$

178. $\int \sin^m x\cos x\,dx = \frac{\sin^{m+1}x}{m+1}.$

179. $\int \cos^m x\sin^n x\,dx = \frac{\cos^{m-1}x\sin^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2}x\sin^n x\,dx.$

180. $\int \cos^m x\sin^n x\,dx = -\frac{\sin^{n-1}x\cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2}x\,dx.$

181. $\int \frac{\cos^m x\,dx}{\sin^n x} = -\frac{\cos^{m+1}x}{(n-1)\sin^{n-1}x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\,dx}{\sin^{n-2}x}.$

182. $$\int \frac{\cos^m x\,dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x\,dx}{\sin^n x}.$$

183. $$\int \frac{\sin^m x\,dx}{\cos^n x} = -\int \frac{\cos^m\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)}.$$

184. $$\int \frac{dx}{\sin^m x \cos^n x}$$
$$= \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1} x\cdot\cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x\cdot\cos^{n-2} x}$$
$$= -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1} x\cdot\cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x\cdot\cos^n x}.$$
$$\int \frac{dx}{\sin x \cos x} = \log \tan x.$$

185. $$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$$

186. $$\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$$

187. $$\int \tan x\,dx = -\log \cos x.$$

188. $$\int \tan^2 x\,dx = \tan x - x.$$

189. $$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

190. $$\int \operatorname{ctn} x\,dx = \log \sin x.$$

191. $$\int \operatorname{ctn}^2 x\,dx = -\operatorname{ctn} x - x.$$

192. $$\int \operatorname{ctn}^n x\,dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x\,dx.$$

193. $$\int \sec x\,dx = \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

194. $$\int \sec^2 x\,dx = \tan x.$$

195. $\int \sec^n x\,dx = \int \frac{dx}{\cos^n x}.$

196. $\int \csc x\,dx = \log \tan \frac{1}{2} x.$

197. $\int \csc^2 x\,dx = -\operatorname{ctn} x.$

198. $\int \csc^n x\,dx = \int \frac{dx}{\sin^n x}.$

199. $\int \frac{dx}{a + b\cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1}\left[\frac{b + a\cos x}{a + b\cos x}\right], \ [a > b > 0],$

or $\frac{1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1}\left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{a + b\cos x}\right], \ [a > b > 0],$

or $\frac{1}{\sqrt{a^2 - b^2}} \cdot \tan^{-1}\left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{b + a\cos x}\right], \ [a > b > 0],$

or $\frac{1}{\sqrt{b^2 - a^2}} \log\left[\frac{b + a\cos x + \sqrt{b^2 - a^2} \cdot \sin x}{a + b\cos x}\right], \ [a > 0, \ b^2 > a^2].$

200. $\int \frac{dx}{a + b\cos x + c\sin x}$

$$= \frac{-1}{\sqrt{a^2 - b^2 - c^2}} \cdot \sin^{-1}\left[\frac{b^2 + c^2 + a(b\cos x + c\sin x)}{\sqrt{b^2 + c^2}(a + b\cos x + c\sin x)}\right]$$

or $\frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log$

$$\left[\frac{b^2 + c^2 + a(b\cos x + c\sin x) + \sqrt{b^2 + c^2 - a^2}(b\sin x - c\cos x)}{\sqrt{b^2 + c^2}(a + b\cos x + c\sin x)}\right].$$

201. $\int x\sin x\,dx = \sin x - x\cos x.$

202. $\int x^2 \sin x\,dx = 2x\sin x - (x^2 - 2)\cos x.$

203. $\int x^3 \sin x\,dx = (3x^2 - 6)\sin x - (x^3 - 6x)\cos x.$

204. $\int x^m \sin x\,dx = -x^m \cos x + m\int x^{m-1}\cos x\,dx.$

205. $\int x \cos x\,dx = \cos x + x \sin x.$

206. $\int x^2 \cos x\,dx = 2x \cos x + (x^2 - 2) \sin x.$

207. $\int x^3 \cos x\,dx = (3x^2 - 6) \cos x + (x^3 - 6x) \sin x.$

208. $\int x^m \cos x\,dx = x^m \sin x - m \int x^{m-1} \sin x\,dx.$

209. $\int \frac{\sin x}{x^m}\,dx = -\frac{1}{m-1} \cdot \frac{\sin x}{x^{m-1}} + \frac{1}{m-1} \int \frac{\cos x}{x^{m-1}}\,dx.$

210. $\int \frac{\cos x}{x^m}\,dx = -\frac{1}{m-1} \cdot \frac{\cos x}{x^{m-1}} - \frac{1}{m-1} \int \frac{\sin x}{x^{m-1}}\,dx.$

211. $\int \frac{\sin x}{x}\,dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \cdots.$

212. $\int \frac{\cos x}{x}\,dx = \log x - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \frac{x^8}{8 \cdot 8!} \cdots.$

213. $\int \sin(mx + a) \cdot \sin(nx + b)\,dx$

$$= \frac{\sin(mx - nx + a - b)}{2(m - n)} - \frac{\sin(mx + nx + a + b)}{2(m + n)}.$$

214. $\int \cos(mx + a) \cdot \cos(nx + b)\,dx$

$$= \frac{\sin(mx + nx + a + b)}{2(m + n)} + \frac{\sin(mx - nx + a - b)}{2(m - n)}.$$

215. $\int \sin(mx + a) \cdot \cos(nx + b)\,dx$

$$= -\frac{\cos(mx + nx + a + b)}{2(m + n)} - \frac{\cos(mx - nx + a - b)}{2(m - n)}.$$

216. $\int \sin(mx + a) \cdot \sin(mx + b)\,dx$

$$= \frac{x}{2} \cdot \cos(b - a) - \frac{\sin(mx + a) \cdot \cos(mx + b)}{2m}.$$

217. $\int \sin(mx + a) \cdot \cos(mx + b)\,dx$

$$= \frac{\sin(mx + a) \cdot \sin(mx + b)}{2m} - \frac{x}{2} \cdot \sin(b - a).$$

218. $\int \cos(mx+a)\cdot\cos(mx+b)\,dx$
$$= \frac{x}{2}\cdot\cos(b-a) + \frac{\sin(mx+a)\cos(mx+b)}{2m}.$$

219. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}.$

220. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2}.$

221. $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2).$

222. $\int \operatorname{ctn}^{-1}x\,dx = x\operatorname{ctn}^{-1}x + \frac{1}{2}\log(1+x^2).$

223. $\int \operatorname{versin}^{-1}x\,dx = (x-1)\operatorname{versin}^{-1}x + \sqrt{2x-x^2}.$

224. $\int (\sin^{-1}x)^2 dx = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1}x.$

225. $\int x\cdot\sin^{-1}x\,dx = \frac{1}{4}\left[(2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}\right].$

226. $\int x^n\sin^{-1}x\,dx = \frac{x^{n+1}\sin^{-1}x}{n+1} - \frac{1}{n+1}\int\frac{x^{n+1}dx}{\sqrt{1-x^2}}.$

227. $\int x^n\cos^{-1}x\,dx = \frac{x^{n+1}\cos^{-1}x}{n+1} + \frac{1}{n+1}\int\frac{x^{n+1}dx}{\sqrt{1-x^2}}.$

228. $\int x^n\tan^{-1}x\,dx = \frac{x^{n+1}\tan^{-1}x}{n+1} - \frac{1}{n+1}\int\frac{x^{n+1}dx}{1+x^2}.$

229. $\int \log x\,dx = x\log x - x.$

230. $\int \frac{(\log x)^n}{x}dx = \frac{1}{n+1}(\log x)^{n+1}.$

231. $\int \frac{dx}{x\log x} = \log(\log x).$

232. $\int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}.$

233. $\int x^m\log x\,dx = x^{m+1}\left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2}\right].$

234. $\int e^{ax}\,dx = \frac{e^{ax}}{a}.$

235. $\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1).$

236. $\int x^m e^{ax}\,dx = \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1}e^{ax}\,dx.$

237. $\int \frac{e^{ax}}{x^m}\,dx = -\frac{1}{m-1}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{e^{ax}}{x^{m-1}}\,dx.$

238. $\int e^{ax}\log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx.$

239. $\int e^{ax}\cdot\sin px\,dx = \frac{e^{ax}(a\sin px - p\cos px)}{a^2+p^2}.$

240. $\int e^{ax}\cdot\cos px\,dx = \frac{e^{ax}(a\cos px + p\sin px)}{a^2+p^2}.$

241. $\int \sinh x\,dx = \cosh x; \quad \int \cosh x\,dx = \sinh x.$

242. $\int \tanh x\,dx = \log\cosh x; \quad \int \operatorname{ctnh} x\,dx = \log\sinh x.$

243. $\int \operatorname{sech} x\,dx = 2\tan^{-1}(e^x).$

244. $\int \operatorname{csch} x\,dx = \log\tanh\left(\frac{x}{2}\right).$

245. $\int x\sinh x\,dx = x\cosh x - \sinh x.$

246. $\int x\cosh x\,dx = x\sinh x - \cosh x.$

247. $\int \cosh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x + x).$

248. $\int \sinh x\cosh x\,dx = \frac{1}{4}\cosh(2x).$

249. $\int \sinh^2 x\,dx = \frac{1}{2}(\sinh x\cosh x - x).$

IV. MISCELLANEOUS DEFINITE INTEGRALS

250. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a>0$; 0, if $a=0$; $-\frac{\pi}{2}$, if $a<0$.

251. $\int_0^\infty x^{n-1}e^{-x}dx = \int_0^1 \left[\log\frac{1}{x}\right]^{n-1}dx = \Gamma(n).$

$\Gamma(n+1) = n\cdot\Gamma(n)$, if $n>0$. $\quad \Gamma(2)=\Gamma(1)=1.$

$\Gamma(n+1)=n!$, if n is an integer. $\quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}.$

$\Gamma(n)=\Pi(n-1). \quad Z(y)=D_y[\log\Gamma(y)].$

$Z(1)=-0.577216.$

252. $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

253. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$= \frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots(n)}\cdot\frac{\pi}{2}$, if n is an even integer;

$= \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdot7\cdots n}$, if n is an odd integer;

$= \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$ for any value of n greater than -1.

254. $\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}$, if $m>0$; 0, if $m=0$; $-\frac{\pi}{2}$, if $m<0$.

255. $\int_0^\infty \frac{\sin x\cdot\cos mx\,dx}{x} = 0$, if $m<-1$ or $m>1$;

$\frac{\pi}{4}$, if $m=-1$ or $m=1$; $\frac{\pi}{2}$, if $-1<m<1$.

256. $\int_0^\infty \frac{\sin^2 x\,dx}{x^2} = \frac{\pi}{2}.$

257. $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

258. $\int_0^\pi \sin kx \sin mx\,dx = \int_0^\pi \cos kx \cos mx\,dx = 0,\ [k \neq m].$

259. $\int_0^\pi \sin kx \cos mx\,dx = \frac{2k}{k^2 - m^2},$ if $k - m$ is odd;

$= 0,$ if $k - m$ is even.

260. $\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}.$

261. $\int_0^\pi \sin kx \cos kx\,dx = 0.$

262. $\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}},\ [a > b > 0].$

263. $\int_0^\infty \frac{\cos mx\,dx}{1 + x^2} = \frac{\pi}{2}\cdot e^{-m}.$

264. $\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$

265. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2\sin^2 x}} = K$

$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right],$ if $k^2 < 1.$

266. $\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2 x}\cdot dx = E$

$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right],$ if $k^2 < 1.$

267. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right).$

268. $\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$

269. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$

270. $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}dx = \frac{e^{-2a}\sqrt{\pi}}{2}.$

271. $\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}$, if $a > 0$.

272. $\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}$, if $a > 0$.

273. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-\frac{b^2}{4a^2}}}{2a}.$

274. $\int_0^1 \frac{\log x}{1-x}dx = -\frac{\pi^2}{6}.$

275. $\int_0^1 \frac{\log x}{1+x}dx = -\frac{\pi^2}{12}.$

276. $\int_0^1 \frac{\log x}{1-x^2}dx = -\frac{\pi^2}{8}.$

277. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}.$

278. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}.$

279. $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$

280. $\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, $[m+1>0,\ n+1>0]$.

281. $\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}} \log\cos x\,dx = -\frac{\pi}{2}\cdot\log 2.$

282. $\int_0^\pi x\cdot\log\sin x\,dx = -\frac{\pi^2}{2}\log 2.$

Natural Logarithms of Numbers between 1.0 and 9.9

N.	0	1	2	3	4	5	6	7	8	9
1.	0.000	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2.	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3.	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.308	1.335	1.361
4.	1.386	1.411	1.435	1.459	1.482	1.504	1.526	1.548	1.569	1.589
5.	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6.	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7.	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8.	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9.	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

Natural Logarithms of Whole Numbers from 10 to 109

N.	0	1	2	3	4	5	6	7	8	9
1	2.303	2.398	2.485	2.565	2.639	2.708	2.773	2.833	2.890	2.944
2	2.996	3.045	3.091	3.135	3.178	3.219	3.258	3.296	3.332	3.367
3	3.401	3.434	3.466	3.497	3.526	3.555	3.584	3.611	3.638	3.664
4	3.689	3.714	3.738	3.761	3.784	3.807	3.829	3.850	3.871	3.892
5	3.912	3.932	3.951	3.970	3.989	4.007	4.025	4.043	4.060	4.078
6	4.094	4.111	4.127	4.143	4.159	4.174	4.190	4.205	4.220	4.234
7	4.248	4.263	4.277	4.290	4.304	4.317	4.331	4.344	4.357	4.369
8	4.382	4.394	4.407	4.419	4.431	4.443	4.454	4.466	4.477	4.489
9	4.500	4.511	4.522	4.533	4.543	4.554	4.564	4.575	4.585	4.595
10	4.605	4.615	4.625	4.635	4.644	4.654	4.663	4.673	4.682	4.691

Values in Circular Measure of Angles which are given in Degrees and Minutes

1′	0.0003	9′	0.0026	3°	0.0524	20°	0.3491	100°	1.7453
2′	0.0006	10′	0.0029	4°	0.0698	30°	0.5236	110°	1.9199
3′	0.0009	20′	0.0058	5°	0.0873	40°	0.6981	120°	2.0944
4′	0.0012	30′	0.0087	6°	0.1047	50°	0.8727	130°	2.2689
5′	0.0015	40′	0.0116	7°	0.1222	60°	1.0472	140°	2.4435
6′	0.0017	50′	0.0145	8°	0.1396	70°	1.2217	150°	2.6180
7′	0.0020	1′	0.0175	9°	0.1571	80°	1.3963	160°	2.7925
8′	0.0023	2′	0.0349	10°	0.1745	90°	1.5708	170°	2.9671

Natural Trigonometric Functions

Angle	Sin	Csc	Tan	Ctn	Sec	Cos	
0°	0.000	∞	0.000	∞	1.000	1.000	90°
1	0.017	57.30	0.017	57.29	1.000	1.000	89
2	0.035	28.65	0.035	28.64	1.001	0.999	88
3	0.052	19.11	0.052	19.08	1.001	0.999	87
4	0.070	14.34	0.070	14.30	1.002	0.998	86
5°	0.087	11.47	0.087	11.43	1.004	0.996	85°
6	0.105	9.567	0.105	9.514	1.006	0.995	84
7	0.122	8.206	0.123	8.144	1.008	0.993	83
8	0.139	7.185	0.141	7.115	1.010	0.990	82
9	0.156	6.392	0.158	6.314	1.012	0.988	81
10°	0.174	5.759	0.176	5.671	1.015	0.985	80°
11	0.191	5.241	0.194	5.145	1.019	0.982	79
12	0.208	4.810	0.213	4.705	1.022	0.978	78
13	0.225	4.445	0.231	4.331	1.026	0.974	77
14	0.242	4.134	0.249	4.011	1.031	0.970	76
15°	0.259	3.864	0.268	3.732	1.035	0.966	75°
16	0.276	3.628	0.287	3.487	1.040	0.961	74
17	0.292	3.420	0.306	3.271	1.046	0.956	73
18	0.309	3.236	0.325	3.078	1.051	0.951	72
19	0.326	3.072	0.344	2.904	1.058	0.946	71
20°	0.342	2.924	0.364	2.747	1.064	0.940	70°
21	0.358	2.790	0.384	2.605	1.071	0.934	69
22	0.375	2.669	0.404	2.475	1.079	0.927	68
23	0.391	2.559	0.424	2.356	1.086	0.921	67
24	0.407	2.459	0.445	2.246	1.095	0.914	66
25°	0.423	2.366	0.466	2.145	1.103	0.906	65°
26	0.438	2.281	0.488	2.050	1.113	0.899	64
27	0.454	2.203	0.510	1.963	1.122	0.891	63
28	0.469	2.130	0.532	1.881	1.133	0.883	62
29	0.485	2.063	0.554	1.804	1.143	0.875	61
30°	0.500	2.000	0.577	1.732	1.155	0.866	60°
31	0.515	1.942	0.601	1.664	1.167	0.857	59
32	0.530	1.887	0.625	1.600	1.179	0.848	58
33	0.545	1.836	0.649	1.540	1.192	0.839	57
34	0.559	1.788	0.675	1.483	1.206	0.829	56
35°	0.574	1.743	0.700	1.428	1.221	0.819	55°
36	0.588	1.701	0.727	1.376	1.236	0.809	54
37	0.602	1.662	0.754	1.327	1.252	0.799	53
38	0.616	1.624	0.781	1.280	1.269	0.788	52
39	0.629	1.589	0.810	1.235	1.287	0.777	51
40°	0.643	1.556	0.839	1.192	1.305	0.766	50°
41	0.656	1.524	0.869	1.150	1.325	0.755	49
42	0.669	1.494	0.900	1.111	1.346	0.743	48
43	0.682	1.466	0.933	1.072	1.367	0.731	47
44	0.695	1.440	0.966	1.036	1.390	0.719	46
45°	0.707	1.414	1.000	1.000	1.414	0.707	45°
	Cos	Sec	Ctn	Tan	Csc	Sin	Angle

Values of the Complete Elliptic Integrals, *K* and *E*, for Different Values of the Modulus, *k*

$$K = \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}; \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

$\sin^{-1} k$	K	E	$\sin^{-1} k$	K	E	$\sin^{-1} k$	K	E
0°	1.5708	1.5708	50°	1.9356	1.3055	81.0°	3.2553	1.0338
1°	1.5709	1.5707	51°	1.9539	1.2963	81.2°	3.2771	1.0326
2°	1.5713	1.5703	52°	1.9729	1.2870	81.4°	3.2995	1.0313
3°	1.5719	1.5697	53°	1.9927	1.2776	81.6°	3.3223	1.0302
4°	1.5727	1.5689	54°	2.0133	1.2681	81.8°	3.3458	1.0290
5°	1.5738	1.5678	55°	2.0347	1.2587	82.0°	3.3699	1.0278
6°	1.5711	1.5665	56°	2.0571	1.2492	82.2°	3.3946	1.0267
7°	1.5767	1.5649	57°	2.0804	1.2397	82.4°	3.4199	1.0256
8°	1.5785	1.5632	58°	2.1047	1.2301	82.6°	3.4460	1.0245
9°	1.5805	1.5611	59°	2.1300	1.2206	82.8°	3.4728	1.0234
10°	1.5828	1.5589	60°	2.1565	1.2111	83.0°	3.5004	1.0223
11°	1.5854	1.5564	61°	2.1842	1.2015	83.2°	3.5288	1.0213
12°	1.5882	1.5537	62°	2.2132	1.1921	83.4°	3.5581	1.0202
13°	1.5913	1.5507	63°	2.2435	1.1826	83.6°	3.5884	1.0192
14°	1.5946	1.5476	64°	2.2754	1.1732	83.8°	3.6196	1.0182
15°	1.5981	1.5442	65°	2.3088	1.1638	84.0°	3.6519	1.0172
16°	1.6020	1.5405	65.5°	2.3261	1.1592	84.2°	3.6853	1.0163
17°	1.6061	1.5367	66.0°	2.3439	1.1546	84.4°	3.7198	1.0153
18°	1.6105	1.5326	66.5°	2.3622	1.1499	84.6°	3.7557	1.0144
19°	1.6151	1.5283	67.0°	2.3809	1.1454	84.8°	3.7930	1.0135
20°	1.6200	1.5238	67.5°	2.4001	1.1408	85.0°	3.8317	1.0127
21°	1.6252	1.5191	68.0°	2.4198	1.1362	85.2°	3.8721	1.0118
22°	1.6307	1.5141	68.5°	2.4401	1.1317	85.4°	3.9142	1.0110
23°	1.6365	1.5090	69.0°	2.4610	1.1273	85.6°	3.9583	1.0102
24°	1.6426	1.5037	69.5°	2.4825	1.1228	85.8°	4.0044	1.0094
25°	1.6490	1.4981	70.0°	2.5046	1.1184	86.0°	4.0528	1.0087
26°	1.6557	1.4924	70.5°	2.5273	1.1140	86.2°	4.1037	1.0079
27°	1.6627	1.4864	71.0°	2.5507	1.1096	86.4°	4.1574	1.0072
28°	1.6701	1.4803	71.5°	2.5749	1.1053	86.6°	4.2142	1.0065
29°	1.6777	1.4740	72.0°	2.5998	1.1011	86.8°	4.2744	1.0059
30°	1.6858	1.4675	72.5°	2.6256	1.0968	87.0°	4.3387	1.0053
31°	1.6941	1.4608	73.0°	2.6521	1.0927	87.2°	4.4073	1.0047
32°	1.7028	1.4539	73.5°	2.6796	1.0885	87.4°	4.4812	1.0041
33°	1.7119	1.4469	74.0°	2.7081	1.0844	87.6°	4.5619	1.0036
34°	1.7214	1.4397	74.5°	2.7375	1.0804	87.8°	4.6477	1.0031
35°	1.7312	1.4323	75.0°	2.7681	1.0764	88.0°	4.7427	1.0026
36°	1.7415	1.4248	75.5°	2.7998	1.0725	88.2°	4.8479	1.0022
37°	1.7522	1.4171	76.0°	2.8327	1.0686	88.4°	4.9654	1.0017
38°	1.7633	1.4092	76.5°	2.8669	1.0648	88.6°	5.0988	1.0014
39°	1.7748	1.4013	77.0°	2.9026	1.0611	88.8°	5.2527	1.0010
40°	1.7868	1.3931	77.5°	2.9397	1.0574	89.0°	5.4349	1.0008
41°	1.7992	1.3849	78.0°	2.9786	1.0538	89.1°	5.5402	1.0006
42°	1.8122	1.3765	78.5°	3.0192	1.0502	89.2°	5.6579	1.0005
43°	1.8256	1.3680	79.0°	3.0617	1.0468	89.3°	5.7914	1.0005
44°	1.8396	1.3594	79.5°	3.1064	1.0434	89.4°	5.9455	1.0003
45°	1.8541	1.3506	80.0°	3.1534	1.0401	89.5°	6.1278	1.0002
46°	1.8691	1.3418	80.2°	3.1729	1.0388	89.6°	6.3504	1.0001
47°	1.8848	1.3329	80.4°	3.1928	1.0375	89.7°	6.6385	1.0001
48°	1.9011	1.3238	80.6°	3.2132	1.0363	89.8°	7.0440	1.0000
49°	1.9180	1.3147	80.8°	3.2340	1.0350	89.9°	7.7371	1.0000

Common Logarithms of Γ(n) for Values of n between 1 and 2.

$$\Gamma(n) = \int_0^{\infty} x^{n-1} \cdot e^{-x} dx = \int_0^1 \left[\log \frac{1}{x}\right]^{n-1} dx.$$

n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$	n	$\log_{10}\Gamma(n)$
1.01	$\bar{1}.9975$	1.21	$\bar{1}.9617$	1.41	$\bar{1}.9478$	1.61	$\bar{1}.9517$	1.81	$\bar{1}.9704$
1.02	$\bar{1}.9951$	1.22	$\bar{1}.9605$	1.42	$\bar{1}.9476$	1.62	$\bar{1}.9523$	1.82	$\bar{1}.9717$
1.03	$\bar{1}.9928$	1.23	$\bar{1}.9594$	1.43	$\bar{1}.9475$	1.63	$\bar{1}.9529$	1.83	$\bar{1}.9730$
1.04	$\bar{1}.9905$	1.24	$\bar{1}.9583$	1.44	$\bar{1}.9473$	1.64	$\bar{1}.9536$	1.84	$\bar{1}.9743$
1.05	$\bar{1}.9883$	1.25	$\bar{1}.9573$	1.45	$\bar{1}.9473$	1.65	$\bar{1}.9543$	1.85	$\bar{1}.9757$
1.06	$\bar{1}.9862$	1.26	$\bar{1}.9564$	1.46	$\bar{1}.9472$	1.66	$\bar{1}.9550$	1.86	$\bar{1}.9771$
1.07	$\bar{1}.9841$	1.27	$\bar{1}.9554$	1.47	$\bar{1}.9473$	1.67	$\bar{1}.9558$	1.87	$\bar{1}.9786$
1.08	$\bar{1}.9821$	1.28	$\bar{1}.9546$	1.48	$\bar{1}.9473$	1.68	$\bar{1}.9566$	1.88	$\bar{1}.9800$
1.09	$\bar{1}.9802$	1.29	$\bar{1}.9538$	1.49	$\bar{1}.9474$	1.69	$\bar{1}.9575$	1.89	$\bar{1}.9815$
1.10	$\bar{1}.9783$	1.30	$\bar{1}.9530$	1.50	$\bar{1}.9475$	1.70	$\bar{1}.9584$	1.90	$\bar{1}.9831$
1.11	$\bar{1}.9765$	1.31	$\bar{1}.9523$	1.51	$\bar{1}.9477$	1.71	$\bar{1}.9593$	1.91	$\bar{1}.9846$
1.12	$\bar{1}.9748$	1.32	$\bar{1}.9516$	1.52	$\bar{1}.9479$	1.72	$\bar{1}.9603$	1.92	$\bar{1}.9862$
1.13	$\bar{1}.9731$	1.33	$\bar{1}.9510$	1.53	$\bar{1}.9482$	1.73	$\bar{1}.9613$	1.93	$\bar{1}.9878$
1.14	$\bar{1}.9715$	1.34	$\bar{1}.9505$	1.54	$\bar{1}.9485$	1.74	$\bar{1}.9623$	1.94	$\bar{1}.9895$
1.15	$\bar{1}.9699$	1.35	$\bar{1}.9500$	1.55	$\bar{1}.9488$	1.75	$\bar{1}.9633$	1.95	$\bar{1}.9912$
1.16	$\bar{1}.9684$	1.36	$\bar{1}.9495$	1.56	$\bar{1}.9492$	1.76	$\bar{1}.9644$	1.96	$\bar{1}.9929$
1.17	$\bar{1}.9669$	1.37	$\bar{1}.9491$	1.57	$\bar{1}.9496$	1.77	$\bar{1}.9656$	1.97	$\bar{1}.9946$
1.18	$\bar{1}.9655$	1.38	$\bar{1}.9487$	1.58	$\bar{1}.9501$	1.78	$\bar{1}.9667$	1.98	$\bar{1}.9964$
1.19	$\bar{1}.9642$	1.39	$\bar{1}.9483$	1.59	$\bar{1}.9506$	1.79	$\bar{1}.9679$	1.99	$\bar{1}.9982$
1.20	$\bar{1}.9629$	1.40	$\bar{1}.9481$	1.60	$\bar{1}.9511$	1.80	$\bar{1}.9691$	2.00	0.0000

$$\left\{\begin{array}{l}\Gamma(z+1) = z \cdot \Gamma(z), \text{ if } z > 0;\ \Gamma(2) = \Gamma(1) = 1; \\ [\Gamma(x) \cdot \Gamma(1-x)] = \pi / \sin \pi x, \text{ if } 1 > x > 0.\end{array}\right\}$$

If the values of an analytic function, $f(x)$, are given in a table for consecutive values of the argument, x, with the constant interval d, and if $h = kd$, where k is any desired fraction,

$$f(a+h) = f(a) + k \cdot \Delta_1 + \frac{k(k-1)}{2!} \cdot \Delta_2 + \frac{k(k-1)(k-2)}{3!} \cdot \Delta_3 + \cdots,$$

where $f(a)$ is any tabulated value.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$ function of x alone

$e^{\int X dx}$ int. factor

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

$e^{\int F(x) dx}$ is int. fac

$Mx + Ny \neq 0$

$\frac{1}{Mx + Ny}$

$Mx - Ny \neq 0$

$\frac{1}{Mx - Ny}$